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BOUNDEDNESS AND ASYMPTOTICS
OF THE GENERALIZED
THEODORUS ITERATION

by

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Boundedness and Asymptotics of the Generalized Theodorus Iteration

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Abstract: A weakly chaotic iteration, called the Generalized Theodorus Iteration, is analyzed with respect to its boundedness and asymptotics. The limit sets, which often are strange attractors, are also considered. Applications are discussed.

Key words: Chaotic iteration, strange attractor.

1. Introduction

In [22], Plato states that his teacher, Theodorus of Cyrene, was the first individual to prove that $\sqrt{3}, \sqrt{5}, \ldots, \sqrt{17}$ were irrational [2, p.95]. Since then many scholars have wondered how he did this with the mathematical tools he had at hand, and, perhaps more puzzling, why he stopped at the number seventeen. An answer which is almost certainly incorrect but which is nonetheless intriguing was given by J. H. Anderhub in 1918 (see [3,5,6,10]). He suggested that Theodorus might have used the following construction. Form the triangle with vertices at $(0,0)$, $(1,0)$, $(1,1)$. Note that the hypotenuse has length $\sqrt{2}$. Starting now at the point $(1,1)$, draw a line segment of unit length perpendicular to the hypotenuse and in the counterclockwise direction; this gives a new triangle with this line as one side, the previous hypotenuse as the other side, and a hypotenuse of length $\sqrt{3}$. Continuing in this way, a spiral figure (called the *quadratwurzelschnecke* [10], or square-root-snail) emerges with the numbers $\sqrt{n}$ as the lengths of the radii (Fig. 1). The last hypotenuse drawn before the figure overlaps itself has length $\sqrt{17}$, and this, it was suggested, was why Theodorus stopped at seventeen (see [5]).

Although this is almost certainly the wrong answer, the sequence of points formed by this process was studied on its own merits by Hlawka ([10]; see also [24]). He proved a number of results concerning this construction. The study was then picked up by Davis [3], who noted that the points could be described in the complex plane by the iteration

$$z_{n+1} = z_n + iz_n/|z_n| \tag{1.1}$$

(with $z_0=1$) and that they were asymptotically an Archimedean spiral. He then interpolated an analytic curve to these points, which he called the Spiral of Theodorus.

Noting that (1.1) could be considered Euler's method for the ordinary differential equation

$$\dot{z} = z/|z|$$

with unit step size, Davis suggested as a generalization of (1.1) the iteration

$$z_{n+1} = \alpha z_n + \beta z_n/|z_n|$$

with $\alpha,\beta\in\mathbb{C}$ and z_0 a given complex number. This iteration, called the Complex Generalized Theodorus Iteration (CGTI), is studied in [3,15,20]. Davis later suggested the further generalization

$$V_{n+1} = A*V_n + B*V_n/\|V_n\| \tag{1.2}$$

where A and B are real mxm matrices, V_0 is a given nonzero m-vector, and $\|\cdot\|$ is the Euclidean vector norm. This iteration, named the Generalized Theodorus Iteration (GTI) by the present author, displays a great variety of strange attractors for appropriate choices of A and B (Figs. 2-5), and includes the complex GTI as a special case; it is studied in [4,15,18,19]. In this report will summarize some of the information contained in [18] and [19] (adding some material from [15]) with respect to sufficient boundedness conditions and asymptotic estimates for this iteration. Since we are interested in boundedness and asymptotics, in stating our results we will ignore the case where V_0 is such that V_n is the null vector for some n (in which case the iteration stops).

2. Boundedness

Let us try to find sufficient conditions for boundedness of the GTI. Taking norms on both sides of (1.2) gives

$$\|V_{n+1}\| \le \|A\| \cdot \|V_n\| + \|B\|$$

Iterating this relation, we find that

$$\|V_n\| \le \|A\|^n \cdot \|V_0\| + \|B\| \cdot \sum_{i=0}^{n-1} \|A\|^i \qquad (2.1)$$

and clearly if $\|A\|<1$ then in the limit the norm is bounded by

$$\|B\|/(1 - \|A\|)$$

Hence $\|A\|<1$ is sufficient for boundedness of the orbit (for any V_0). From this and the similarity of (1.2) to the linear system

$$V_{n+1} = A*V_n \qquad (2.2)$$

which has a bounded solution for every V_0 when the spectral radius of A is less than unity (and at least one unbounded solution when the spectral radius of A is greater than unity), it seems reasonable to conjecture that $\rho(A)<1$ suffices for the boundedness of (1.2), where $\rho(A)$ is the spectral radius of A. That this is in fact the case follows from two lemmas. We omit the rather technical proof of Lemma 1. It is not hard to establish this result by using an approach similar to that used in sections five and six but it is rather tedious to carry out the proof in detail (see [15]). Lemma 2 is a discrete version of an ordinary differential equations result found in [1].

Lemma 1: If $\|V_n\|$ is unbounded then $\|V_n\|\longrightarrow\infty$.

Lemma 2: Let A be a matrix with $\rho(A)<1$ and suppose that $B(n)$ is a sequence of matrices with $\|B(n)\|\longrightarrow 0$ as $n\longrightarrow\infty$. Then all solutions of

$$z_{n+1} = (A + B(n))\cdot z_n$$

tend to zero as $n\longrightarrow\infty$.

Proof: Since $\rho(A)<1$ there exists a matrix norm $\|\cdot\|_A$ such that $\|A\|_A<1$. We will choose this to be a natural matrix norm and will also denote by $\|\cdot\|_A$ the vector norm which induces it. Let ϵ equal $(1-\|A\|_A)$. Now since all matrix norms are equivalent in the sense that if $\|M(n)\|_\alpha\longrightarrow 0$ then $\|M(n)\|_\beta\longrightarrow 0$, we have from our hypothesis that $\|B(n)\|_A\longrightarrow 0$. Hence we can choose an N such that $\|B(i)\|_A<\epsilon/2$ for all $i\geq N$.

Now consider iterating

$$z_{n+1} = (A + B(n))\cdot z_n$$

(for some z_0) N times to produce the vector z_N. Set $\omega_0=z_N$ and

$$\omega_{n+1} = (A + B(N + n))\cdot\omega_n$$

$\forall n\geq 0$. Clearly $\omega_i=z_{N+i}$ for all $i\geq 0$. Now

$$\omega_{n+1} = (A + B(N + n))* \cdots *(A + B(N))*\omega_0$$

and so

$$\begin{aligned}\|\omega_{n+1}\|_A &\leq \|A + B(N + n)\|_A * \cdots *\|A + B(N)\|_A *\|\omega_0\|_A \\ &= \|\omega_0\|_A \prod_{i=0}^{n} \|A + B(N + i)\|_A\end{aligned}$$

Then

$$\begin{aligned}\|\omega_{n+1}\|_A &\leq \|\omega_0\|_A \prod_{i=0}^{n} (\|A\|_A + \|B(N + i)\|_A) \\ &\leq \|\omega_0\|_A \prod_{i=0}^{n} ((1 - \epsilon) + \epsilon/2) \\ &= \|\omega_0\|_A \prod_{i=0}^{n} (1 - \epsilon/2)\end{aligned}$$

which clearly tends to zero as $n\to\infty$. Hence $\|\omega_n\|_A\to 0$ as $n\to\infty$ so that $\|z_n\|_A\to 0$ as $n\to\infty$, and so by the equivalence of vector norms on $\mathbb{R}^m$ we have that $\|z_n\|\to 0$ as $n\to\infty$. This completes the proof. □

We may now show the desired result.

Theorem 1: If $\rho(A)<1$ then the solution of (1.2) is bounded.

Proof: Suppose the contrary and choose a V_0 such that $\|V_n\|$ is unbounded. By Lemma 1, $\|V_n\|\to\infty$. Define $B(n)=B/\|V_n\|$. By the hypothesis, $\|B(n)\|\to 0$. Consider the iteration

$$z_{n+1} = (A + B(n))*z_n$$

with $z_0=V_0$. Clearly $z_n=V_n$ for all n. The conditions of Lemma 2 are met and so $\|V_n\|\to 0$ as $n\to\infty$. But this contradicts $\|V_n\|$ unbounded, which establishes the result. □

It is also possible to show that if $\rho(A)>1$ then there exists a V_0 such that $\|V_n\|$ is unbounded (see section 5). In this sense the GTI is surprisingly similar to the linear system (2.2).

In the particular case A=0 a detailed analysis is possible [16,17]. This case is related to the power method for the numerical solution of eigenvalue problems.

3. Asymptotics

This leaves the question of boundedness for the case $\rho(A)=1$. It is possible to construct examples showing that when A is unispectral the orbits of the GTI may be always bounded for one choice of B and always unbounded for another choice of B. However, a general statement about the asymptotics of the iteration may be made when A is of bounded type, that is, when $\rho(A)\le 1$ and any eigenvalues of A that lie on the unit circle are simple (i.e. have as many linearly independent eigenvectors as their multiplicity). In this case the linear system (2.2) has the property that all solutions are bounded (although in general the bound will depend on V_0). Before stating the corresponding result for the GTI, we state:

Hukuwara's Theorem: Let z_t be any solution of

$$z_{t+1} = (A + B(t))*z_t \tag{3.1}$$

for t=0,1,2,... where A is a constant matrix and

$$\sum_{t=1}^{\infty} \|B(t)\| < \infty$$

Suppose that A is of bounded type, so that all solutions of $y_t=A*y_{t-1}$ are bounded for $t\ge 0$. Then every solution of (3.1) is bounded.

This result, an analogue of a theorem from differential equations, may be found in [21]. The proof there uses the ℓ_1 vector norm, but the argument is valid for any vector norm and compatible matrix norm. We will now use it to prove an asymptotic result for the GTI.

Theorem 2: Let A be of bounded type in (1.2). Then

$$\sum_{n=0}^{\infty} 1/\|V_n\| = \infty$$

Proof: Suppose the contrary, i.e. that for some V_0 we have

$$\sum_{n=0}^{\infty} 1/\|V_n\| < \infty \tag{3.2}$$

and construct such a sequence $\{V_n\}$ by using this V_0 and (1.2). Define $B(n)=B/\|V_n\|$ and consider the iteration

$$Y_{n+1} = (A + B(n))*Y_n$$

where we set $Y_0=V_0$. Clearly $Y_n=V_n$ for all $n\ge 0$; the two sequences are identical. Now

$$\sum_{n=0}^{\infty} \|B(n)\| = \sum_{n=0}^{\infty} \|B\|/\|V_n\|$$
$$= \|B\| \sum_{n=0}^{\infty} 1/\|V_n\|$$

which by supposition is bounded. Therefore by Hukuwara's Theorem, $\|Y_n\|$ is bounded. Then $\|V_n\|$ is bounded, that is, $\exists M>0$ such that $\|V_n\|<M$ for all n. But then $\|1/V_n\|>1/M$ and since $1/\|V_n\| \to 0$ we cannot have (3.2). This contradiction establishes the result. □

In fact a slightly more general result holds. If $\|\cdot\|$ is any vector norm and $\Phi(\cdot)$ is any function that is continuous on $[0,\infty)$ and positive on $(0,\infty)$, and if A is of bounded type, then any solution of

$$V_{n+1} = A*V_n + B*V_n/\Phi(\|V_n\|)$$

must be such that

$$\sum_{n=0}^{\infty} 1/\Phi(\|V_n\|) = \infty$$

This may be shown in a completely analogous manner (see [15] for details and applications).

Theorem 2 establishes that, when $\rho(A)=1$ but A is of bounded type, if the orbits diverge then they do so in a way that is not much worse than linear (since the sum of the reciprocals of the norms diverges). In fact if $\|A\|=1$ in the spectral norm then it is a fact that the divergence is at most linear, as can be seen by setting $\|A\|=1$ in (2.1) to get

$$\|V_n\| \le \|V_0\| + n\|B\|$$

as the asymptotic estimate. We will later consider this point in greater detail when we show in section 5 that if A is of bounded type then the divergence is always at most linear.

4. Green's Function Representation

In order to refine our asymptotic estimates of the previous section, we will need a formula for the solution of the iteration (1.2). Given an iteration of the form

$$z_t = A(t)*z_{t-1} + \omega_{t-1}$$

with ω_t and z_0 given vectors and $A(t)$ a time-varying matrix, the solution can always be written in the form

$$z_{t+1} = \sum_{s=0}^{t} Y_t \cdot Y_s^{-1} \cdot \omega_s + Y_{t+1} \cdot Y_0^{-1} \cdot c$$

where c is the initial condition z_0, and $\{Y_t\}$ is a fundamental matrix set of solutions for

$$z_t = A(t)*z_{t-1} \tag{4.1}$$

(that is, $Y_0 = I$ and $Y_{n+1} = A(n)*Y_n$) [21]. For the GTI, the matrix A is constant, and $z_0 = V_0$ is given. Furthermore, $Y_n = A^n$ for the linear system (2.2) corresponding to (4.1), and

$$\omega_t = B*V_t / \|V_t\|$$

so that

$$V_{n+1} = A^{n+1} V_0 + \sum_{t=0}^{n} A^{n-t} * B * V_t / \|V_t\| \tag{4.2}$$

is the solution of (1.1). This is called the Green's Function Representation of the solution. We will use this form of the GTI to address the questions of boundedness and asymptotics.

5. A Direct Approach to Boundedness

The approach used in the second section to show that $\rho(A)<1$ suffices for the boundedness of (1.2) is the one originally employed by the author in [15] and also appears in [18]. A more direct method, however, was used by the author in [19], and we now give this method.

Suppose that $\rho(A)<1$. Then there exists some natural matrix norm $\|\cdot\|_\alpha$ such that $\|A\|_\alpha < 1$. Using $\|\cdot\|_\alpha$ to represent the vector norm which induces this matrix norm as well, we have from (4.2) that

$$\|V_{n+1}\|_\alpha \le \|A^{n+1}V_0\|_\alpha + \left\| \sum_{t=0}^{n} A^{n-t}*B*V_t/\|V_t\| \right\|_\alpha$$

$\forall n\ge 0$. Thus

$$\|V_{n+1}\|_\alpha \le \|A\|_\alpha^{n+1}\cdot\|V_0\|_\alpha + \sum_{t=0}^{n} \|A\|_\alpha^{t}\cdot\|B\|_\alpha\cdot k$$

where k is a positive constant such that

$$\|V\|_\alpha/\|V\| \le k$$

for all nonzero vectors V. Such a k exists since all vector norms are equivalent on $\mathbb{R}^m$ [12]. Summing the geometric series and collecting like terms gives

$$\|V_{n+1}\|_\alpha \le \epsilon(n) + k\cdot\|B\|_\alpha/(1 - \|A\|_\alpha)$$

where $\epsilon(n) \to 0$ as $n\longrightarrow\infty$. Hence $\|V_n\|_\alpha$ is asymptotically bounded, and by the equivalence of vector norms so is $\|V_n\|$. An asymptotic bound for $\|V_n\|_\alpha$ is given by

$$k\cdot\|B\|_\alpha/(1 - \|A\|_\alpha)$$

in terms of the α-norm. When the spectral norm of A is less than unity the corresponding bound using that norm,

$$\|B\|/(1 - \|A\|),$$

is generally fairly tight.

We have shown that $\rho(A)<1$ suffices for boundedness. In an analagous way we can show that if $\rho(A)>1$ then there must exist some V_0 such that $\|V_n\|$ is unbounded. (A similar approach has been employed by W.-F. Chuan; see [15]). For, considering (4.2), we have that

$$\|V_{n+1}\| \geq \left| \|A^{n+1}V_o\| - \left\| \sum_{t=0}^{n} A^{n-t} * B * V_t / \|V_t\| \right\| \right|$$

(using the spectral norm). Let λ be an eigenvalue of A such that $|\lambda| = \rho(A)$ and choose V_o to be an eigenvector of A associated with the eigenvalue λ. Then we have

$$\|V_{n+1}\| \geq \left| \|\lambda^{n+1}V_o\| - \left\| \sum_{t=0}^{n} A^{n-t} * B * V_t / \|V_t\| \right\| \right|$$

and since the norm of the sum is at most the sum of the norms,

$$\begin{aligned} \|V_{n+1}\| &\geq \left| |\lambda|^{n+1}\|V_o\| - \sum_{t=0}^{n} \|A\|^t \cdot \|B\| \right| \\ &\geq \left| \rho_o|\lambda|^{n+1} - \|B\| \cdot (1 - \|A\|^{n+1})/(1 - \|A\|) \right| \end{aligned}$$

with $\rho_o = \|V_o\|$. If we let

$$c = \|B\|/(\|A\| - 1)$$

then $c>0$ and

$$\|V_{n+1}\| \geq \left| \rho_o|\lambda|^{n+1} - c\|A\|^{n+1} + c \right|$$

in terms of c. Now $\rho(A) = |\lambda| + \epsilon$ for some $\epsilon \geq 0$, so

$$\begin{aligned} \|V_{n+1}\| &\geq \left| \rho_o|\lambda|^{n+1} - c(|\lambda| + \epsilon)^{n+1} + c \right| \\ &\geq \left| |\lambda|^{n+1}(\rho_o - c(1 + \epsilon/|\lambda|)^{n+1} + c \right| \qquad (5.1) \end{aligned}$$

If ϵ is equal to zero, we may now choose ρ_o to be greater than c in order to have $\|V_n\| \longrightarrow \infty$. Otherwise, the term involving ϵ will grow arbitrarily large, again giving $\|V_n\| \longrightarrow \infty$.

Again, we remark that the GTI (1.1) has boundedness properties similar to those of the linear system (2.2). In the next section we will derive improved asymptotics for the indeterminate case where A has unit spectral radius. Before doing so let us collect the information above as a theorem.

Theorem 3: Consider the iteration (1.2). If $\rho(A)<1$ then the iteration is bounded for any V_o, and if $\rho(A)>1$ then there exists a V_0 such that the iteration is unbounded.

6. Improved Asymptotic Estimates

From (5.1) it is immediately apparent that when $\rho(A)>1$ the divergence will, in general, be exponential. This again leaves the case $\rho(A)=1$. As before, let us first look at the case where A is of bounded type. Using the spectral matrix norm, (4.2) gives

$$\|V_{n+1}\| \le \|A^{n+1}V_0\| + \left\| \sum_{t=0}^{n} A^{n-t}*B*V_t/\|V_t\| \right\| \qquad (6.1)$$
$$\le K_1 + \sum_{t=0}^{n} \|A^t\|\cdot\|B\|$$

where $K_1 \ge 0$ is a bound on $\|A^{n+1}V_0\|$ (which in general will depend on V_0). Now since A is of bounded type, $\|A^n\|$ is also bounded, that is, $\|A^n\| \le K$ for some $K>0$. Thus

$$\|V_{n+1}\| \le K_1 + \|B\| \sum_{t=0}^{n} K$$
$$\le K_1 + K_2 n$$

showing that the divergence is no worse than linear.

If $\rho(A)=1$ but A is not of bounded type then it is possible to show that $\|A^n\| = \mathcal{O}(n^m)$ (in fact $\|A^n\| = \mathcal{O}(n^{p-1})$ where $p \le m$ is the maximal degree of all nonlinear divisors of A associated with unimodular eigenvalues), where A is m×m. Hence from (6.1) we have

$$\|V_{n+1}\| \le \|A^{n+1}\|\cdot\|V_0\| + \sum_{t=0}^{n} \|A^t\|\cdot\|B\|$$

and clearly $\|V_{n+1}\|$ diverges at most as $\mathcal{O}(n^{m+1})$. (In fact, it diverges as $\mathcal{O}(n^p)$, and hence in the worst case as $\mathcal{O}(n^m)$). Hence, when $\rho(A)=1$, if the iteration diverges then it does so at most polynomially. We write this as a theorem.

Theorem 4: Consider (1.2) and suppose that $\rho(A)=1$ and the iteration is unbounded. If all eigenvalues of A with unit modulus are simple then the divergence is no worse than linear. Otherwise the divergence is no worse than polynomial of order m, where m is the order of A.

7. Limit Sets

Let us define $\Gamma(A,B,V_0)$ to be the limit set (or ω-limit set [13]) of the iteration; that is, $Y\in\Gamma(A,B,V_0)$ if there exists an increasing subsequence $n(i)$ such that $V_{n(i)}\longrightarrow Y$ as $i\longrightarrow\infty$ when the initial condition is V_0. Two theorems follow easily from the material in [13].

Theorem 5: Suppose that for some V_0 the sequence $\{V_n\}$ is bounded. Then $\Gamma(A,B,V_0)$ is a non-empty compact set and $V_n\longrightarrow\Gamma(A,B,V_0)$ as $n\longrightarrow\infty$.

Proof: Clearly the closure of Γ is simply Γ and so Γ is closed. The boundedness of $\{V_n\}$ implies that Γ is also bounded. Hence Γ is compact (by definition). By the Bolzano-Weierstrass Theorem [23], Γ is not empty.

Now $V_n\longrightarrow\Gamma$ means that $\inf(\|V_n-Y\|:Y\in\Gamma)\longrightarrow0$ as $n\longrightarrow\infty$ [13]. Suppose the contrary. Then we can find a subsequence of $\{V_n\}$, indexed by, say, $n(i)$, which remains a finite distance from Γ. Now $V_{n(i)}$ is a bounded infinite sequence and so by the Bolzano-Weierstrass Theorem it contains a limit point. But that point must be in Γ, which is a contradiction. □

Theorem 6: Suppose that for some V_0 the sequence $\{V_n\}$ is bounded and $O \notin \Gamma(A,B,V_0)$. Then $\Gamma(A,B,V_0)$ is positively invariant.

Proof: Define

$$T(V) = A*V + B*V/\|V\| \tag{7.1}$$

so that

$$V_{n+1} = T(V_n)$$

is the iteration under consideration. Since $O \notin \Gamma$, the transformation T is continuous in some neighborhood of any point $Y \in \Gamma$. Fix Y and choose a subsequence n(i) such that $V_{n(i)} \longrightarrow Y$. By the continuity of T, we have that

$$T(V_{n(i)}) \longrightarrow T(Y)$$

as $i \longrightarrow \infty$. But

$$T(V_{n(i)}) = V_{n(i)+1}$$

so that

$$V_{m(i)} \longrightarrow T(Y)$$

where $m(i) = n(i)+1$. Hence $T(Y) \in \Gamma$, and Γ is positively invariant. □

In fact, it is clear from the above that $T(Y) \in \Gamma$ whenever $Y \in \Gamma$ and $T(Y)$ is defined. In other words, $\Gamma \setminus \{O\}$ is positively invariant. Further topological results on discrete dynamical systems that can be extended to this case (with some care taken near the origin) may be found in [14].

Suppose that $Y \in \Gamma(A,B,V_0)$, so that $V_{n(i)} \longrightarrow Y$ for some increasing subsequence n(i). Since the transformation $T: \mathbb{R}^m \longrightarrow \mathbb{R}^m$ of (7.1) has the property that

$$T(-V) = -T(V)$$

it follows that the sequence generated by $-V_0$ is $-V_{n(i)}$, which tends to $-Y$; that is, $-Y \in \Gamma(A,B,-V_0)$. In a similar way, it is easy to see that $-Y \in \Gamma(-A,-B,V_0)$ and $Y \in \Gamma(-A,-B,-V_0)$.

In the case of a global attractor, $\Gamma(A,B,V_0)$ is independent of V_0 for the particular A and B under consideration. In this case, it follows from the above that the attractor is centrally symmetric, i.e. $Y \in \Gamma$ implies $-Y \in \Gamma$, and that the same attractor is found when A and B are replaced by -A and -B, respectively. Some of the attractors displayed by this iteration for various choices of A and B are shown in the figures; see also [4, 15]. Iserles has found an analytical expression for the attractors for a certain set of cases of the form $A=\alpha B$, $\alpha>0$ (see [4]). For the case A=0 the form of the attractor is also known [15,16,17].

8. Discussion

The Generalized Theodorus Iteration is in some ways very similar to the simple linear system (Theorem 3) and yet it contains some very complicated structures (Figs. 2-5). It is likely that the attractors are, in general, strange but not always chaotic (see [9]); some computational evidence to this effect has been provided by James Heyman of the Naval Postgraduate School in his study of the use of chaotic discrete dynamical systems, particularly the Hénon attractor, as pseudorandom number generators (personal communication).

Further work on the applications of the GTI in this important area are in progress. Due to the present plethora of definitions of the terms "chaos" and "fractal" (see [15]) it is difficult to prove anything substantative about the chaotic nature of the attractors but, as noted in [11], many well-known strange attractors are still awaiting analytical verification of the observed qualitative features of the underlying iterations and differential systems. One aspect of the GTI which we feel is particularly relevant to the field is the vast array of geometrically distinct attractors to be found in the GTI for different choices of the matrices A and B; even though these two matrices essentially represent eight parameters, the number of distinct attractors is still surprising, and we are not aware of another discrete planar map with this property. Further investigation of this peculiarity seems warranted. Hopefully the figures cited here and in [15] will motivate research on the question of the number of distinct attractors which can belong to a single iteration, as the works by Gleick [7], Devaney [8], and Wiggins [25,26] have served to help motivate and inform this author.

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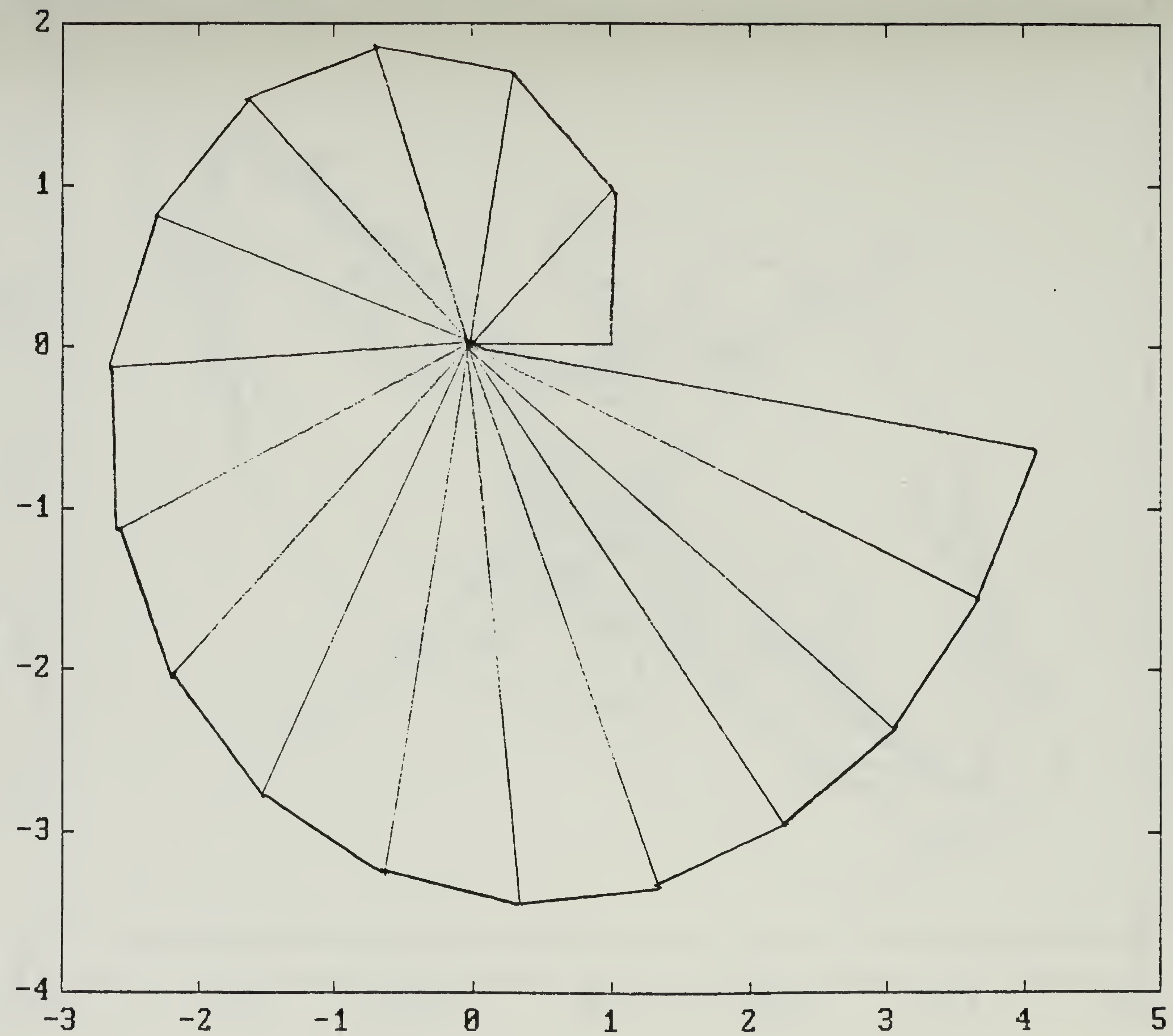
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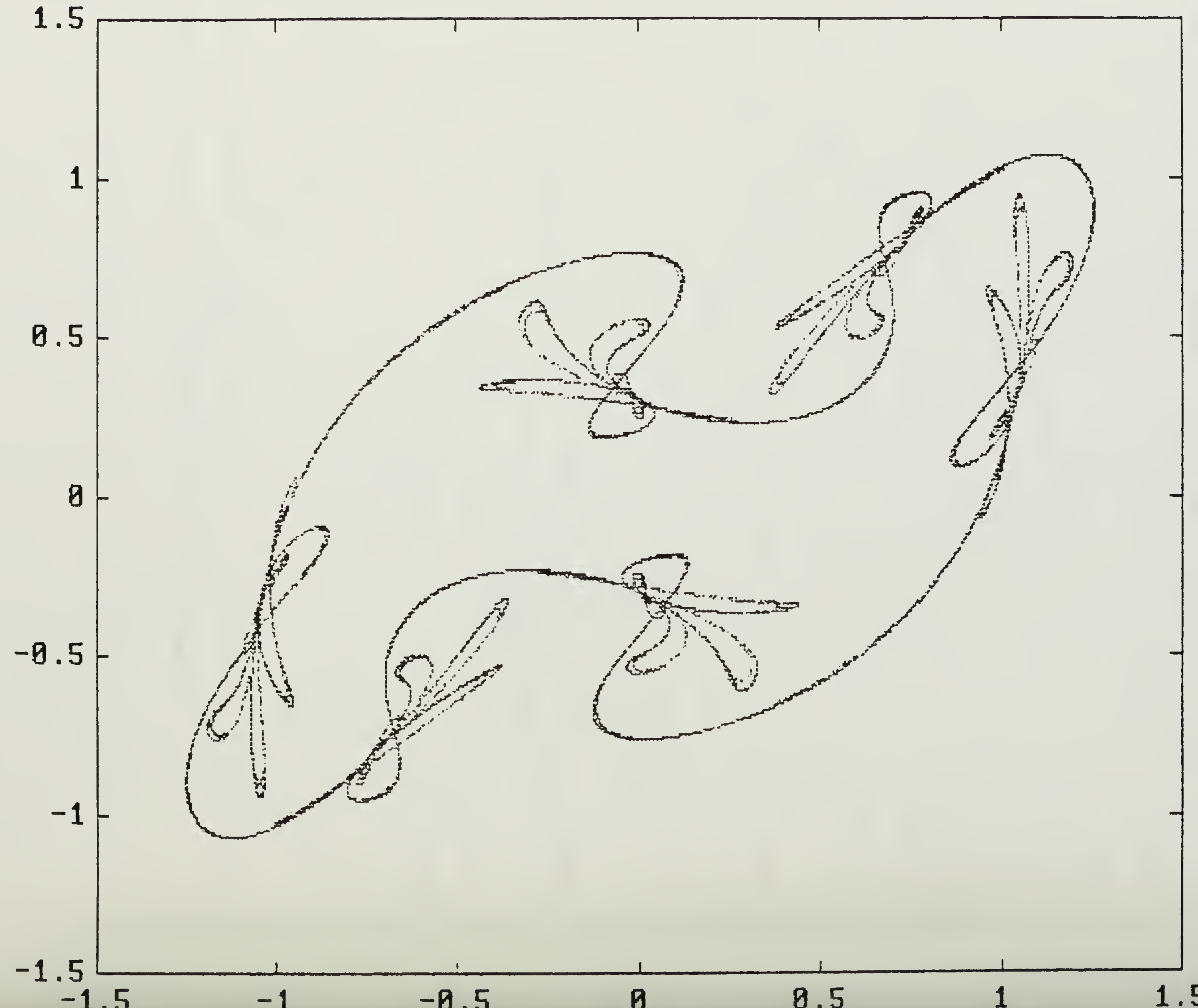
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-0.5
-1
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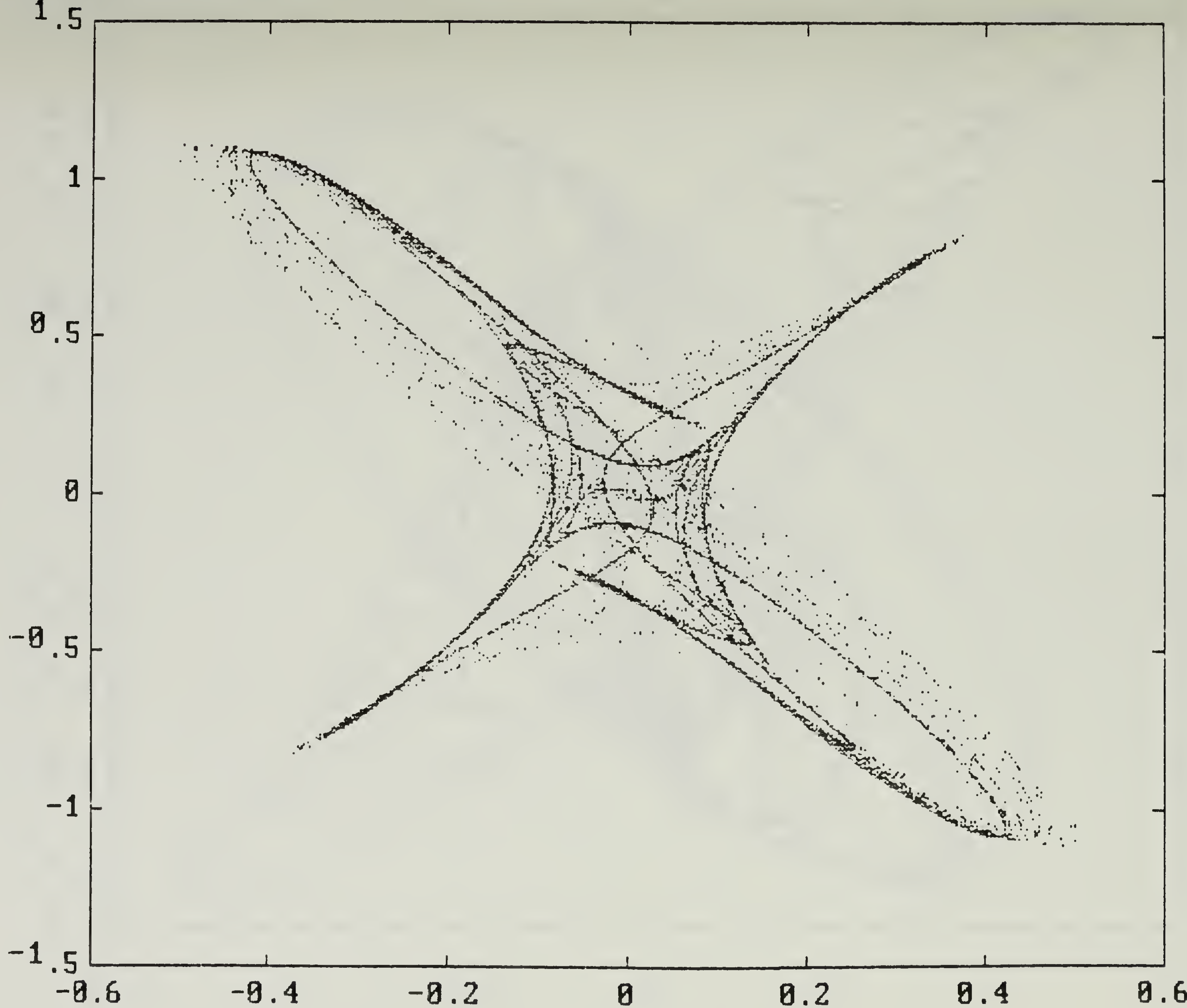
1
1.5



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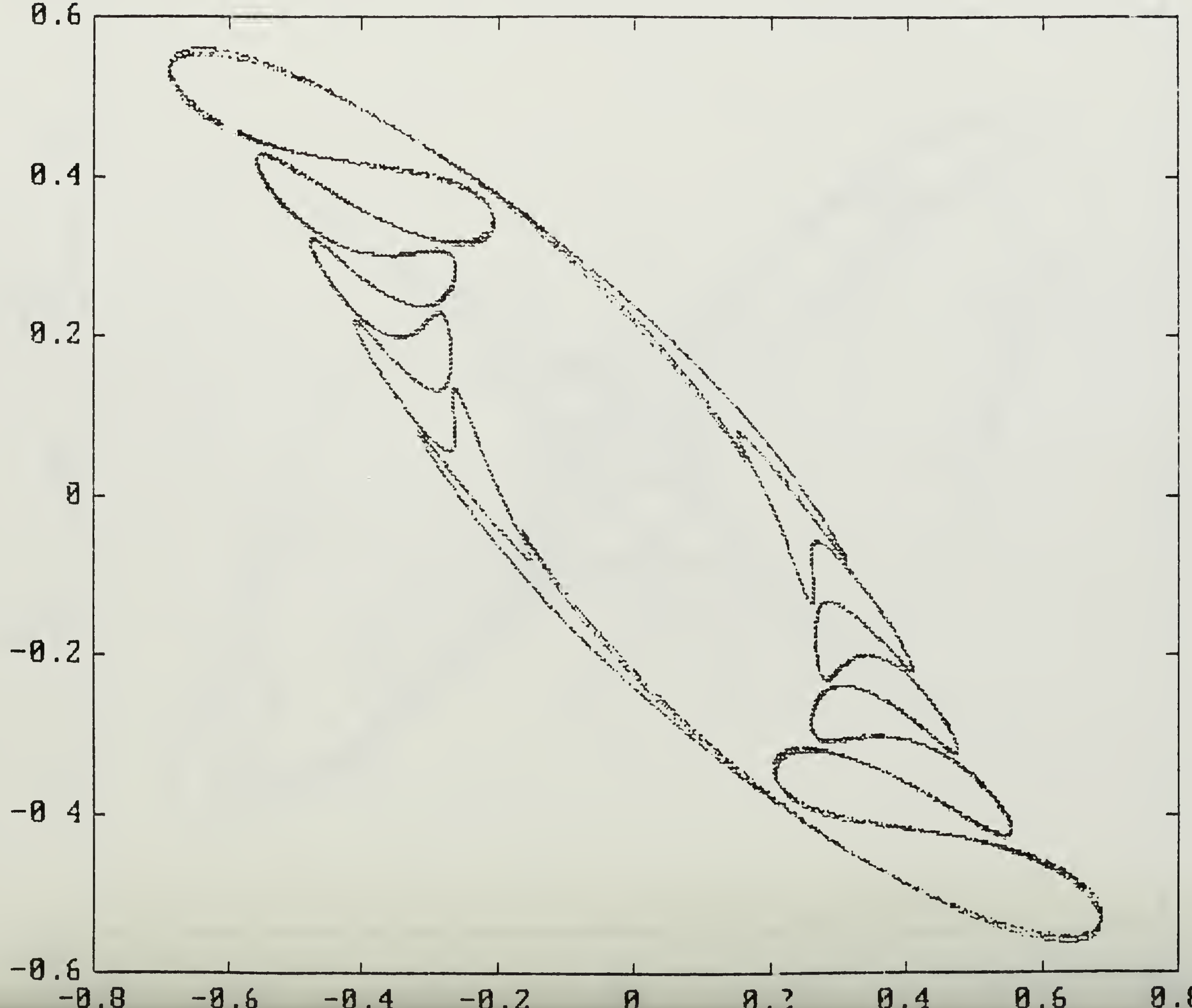
0.2
0.4
0.6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0.4
0.6
0.8

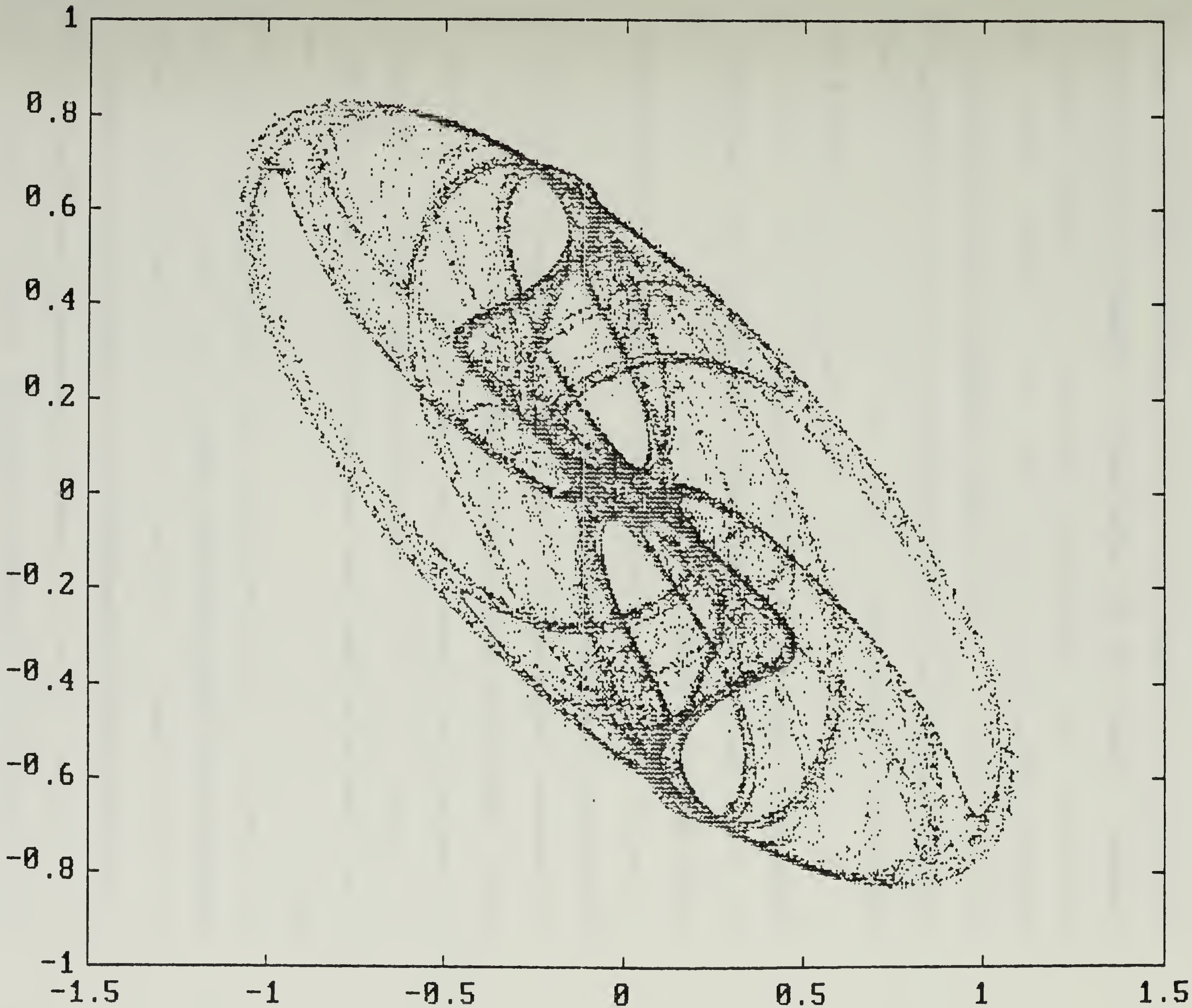

Fig 5: A=[.6228 1.5603;-.23 -.131] B=[-.5577 -.9707; .8056 .2244]

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